01 - 03 April, 2020
08 - 09 April, 2020
Week - 3, 4

The Fibonacci and Related Sequences

Defn: Fibonacci Sequence: It is the series of numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots\ldots$$

The next number is found by adding up the two numbers before it; that is, Fibonacci numbers satisfy the recursion formula

$$F_n + F_{n+1} = F_{n+2}$$

~~Asst.~~ $F_1 = 1 \;;\; F_2 = 1$

For instance:
$$F_3 = F_1 + F_2$$
$$= 1 + 1 = 2$$
$$F_4 = F_2 + F_3 = 1 + 2 = 3$$
$$F_8 = F_7 + F_6$$
$$F_8 = 8 + 13 = 21$$

Defn: Golden Ratio: Robert Simson showed that the ratio of one Fibonacci number to one preceding it

$$\left(\text{i.e. } \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \ldots\ldots \right)$$

draws progressively nearer, alternatively from above and from below, to the golden number ϕ which is given by

$$\phi = \frac{1}{2}(\sqrt{5} + 1) = 1.61803\ldots\ldots$$

Que. Show that the sequence of ratio of Fibonacci Number to the one preceding it ~~converges~~ converges to the golden ratio.

Sol: Since we know that Fibonacci numbers are given by the recursion formula

$$F_n + F_{n+1} = F_{n+2}$$

$$F_1 = 1 \ , \ F_2 = 1$$

and the sequence of fibonacci numbers known as Fibonacci sequence is given by

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Fibonacci Number in the sequence (A)	Next Fibonacci Number (B)	Ratio (B/A)
1	1	1
2	3	$\frac{3}{2} = 1.5$
3	5	1.66666....
5	8	1.6
8	13	1.625
⋮	⋮	⋮
144	233	1.618055.....
233	377	1.618025.....
.....	.....	.....

Since the golden ratio is given by $\phi = \frac{1}{2}(\sqrt{5}+1)$

$$= 1.618034\ldots$$

Clearly, the sequence of ratio of Fibonacci number to the one preceding it converges to the golden ratio.

Lucas Sequence: Ascending integer powers of the golden number yield the following sequence:

$\phi = \frac{1}{2}(\sqrt{5}+1)$ $\qquad \phi^4 = \frac{1}{2}(3\sqrt{5}+7)$

$\phi^2 = \frac{1}{2}(\sqrt{5}+3)$ $\qquad \phi^5 = \frac{1}{2}(5\sqrt{5}+11)$

$\phi^3 = \frac{1}{2}(2\sqrt{5}+4)$ $\qquad \phi^6 = \frac{1}{2}(8\sqrt{5}+18)$ - - - - -

where the coefficients of the irrational $\sqrt{5}$ terms form the Fibonacci sequence and those of rational terms form the Lucas sequence

$$1, 3, 4, 7, 11, 18, 29, 47, \ldots$$

whose terms satisfy the recursion formula

$$L_n + L_{n+1} = L_{n+2}$$
$$L_1 = 1 \;, L_2 = 3$$

Note: Like the Fibonacci sequence, every term of the Lucas sequence is the sum of the two immediately preceding ones.

Using the Golden ratio to calculate Fibonacci Numbers:

The general terms of the Fibonacci and Lucas ~~sequence~~ sequences are given to the Binet formulas for F_n and L_n

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

where $\frac{1+\sqrt{5}}{2}$ is the golden number

Or $F_n = \frac{1}{\sqrt{5}}\left[\phi^n - (1-\phi)^n\right]$

and $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$

Pell number Sequence: The Pell number sequence

$P_n: 1, 2, 5, 12, 29, 70, 169, \ldots$

named after the English mathematician John Pell and

$Q_n: 1, 3, 7, 17, 41, 99, 239, \ldots$

satisfy the recursion formulas

$$\left.\begin{array}{l} P_n + 2 \cdot P_{n+1} = P_{n+2} \\ P_1 = 1;\ P_2 = 2 \end{array}\right\}$$

$$\left.\begin{array}{l} Q_n + 2 \cdot Q_{n+1} = Q_{n+2} \\ Q_1 = 1;\ Q_2 = 3 \end{array}\right\}$$

and the Binnet formulas for the same are given by

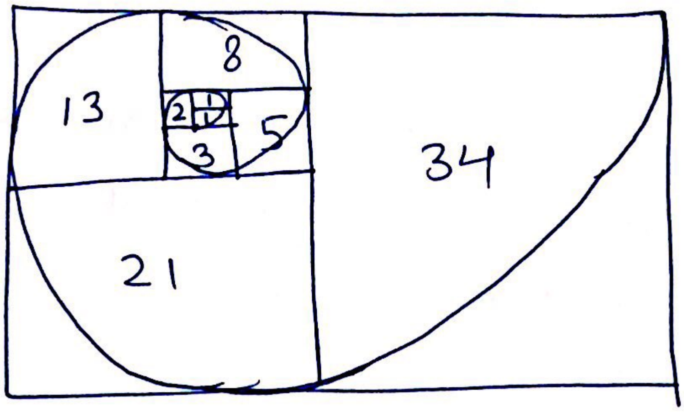
$$P_n = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^n - (1-\sqrt{2})^n\right]$$

$$Q_n = \frac{1}{2}\left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right]$$

Significance of Golden ratio in nature:

Note: Fibonacci numbers and related number sequences appear as natural phenomena, such as the shape of snail shells and the head of sunflowers and in phyllotaxy.

When we make squares with the widths of Fibonacci numbers, we get a nice spiral



Squares are fitted so nicely like 5+8 =13 , new square
and 8 +13 = 21 , new square
and 21 +13 = 34 , new square

and This spiral is found in nature in the heads of sunflowers.

Golden Section

A line segment that is divided into two segments, a greater 'a' and a smaller 'b' such that the length of a+b is to a as a is to b i.e

$$\frac{a+b}{a} = \frac{a}{b}$$

is divided in the golden section or golden ratio.

$$\frac{a+b}{a} = \frac{a}{b}$$

Multiplying both sides by a/b, we get

$$\frac{a}{b}\left(\frac{a+b}{a}\right) = \left(\frac{a}{b}\right)\left(\frac{a}{b}\right)$$

$$\Rightarrow \quad \frac{a}{b} + 1 = \left(\frac{a}{b}\right)^2$$

$$\Rightarrow \left(\frac{a}{b}\right)^2 - \frac{a}{b} - 1 = 0$$

Solving for a/b, we get

$$\frac{a}{b} = \frac{\sqrt{5}+1}{2} \quad \approx \cancel{2}$$

$$= \frac{\sqrt{5}+1}{2} \times \frac{\sqrt{5}-1}{\sqrt{5}-1}$$

$$= \frac{5-1}{2(\sqrt{5}-1)} = \frac{4}{2(\sqrt{5}-1)} = \frac{2}{\sqrt{5}-1}$$

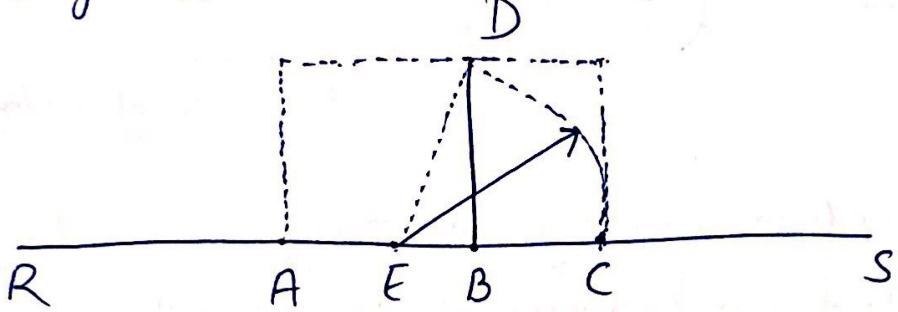
i.e $\quad \dfrac{a}{b} = \dfrac{\sqrt{5}+1}{2} = \dfrac{2}{\sqrt{5}-1}$

Note: i) The golden section is also known as the divine section.

ii) The golden section was a recognized aesthetic guide in art, even a decree absolute, governing the shape and disposition of drawings and paintings and in architecture, where the facades of many public edifices were proportioned according to this ratio.

Construction of a golden Section:

Given: on a straight line RS, a line segment AB of length a.



1.) Construct at B a perpendicular BD of length a

2) Bisect AB at E; with E as center and ED as radius, describe an arc that intersects RS at C.

By Pythagorean theorem, $r = \sqrt{5}\,\frac{a}{2}$

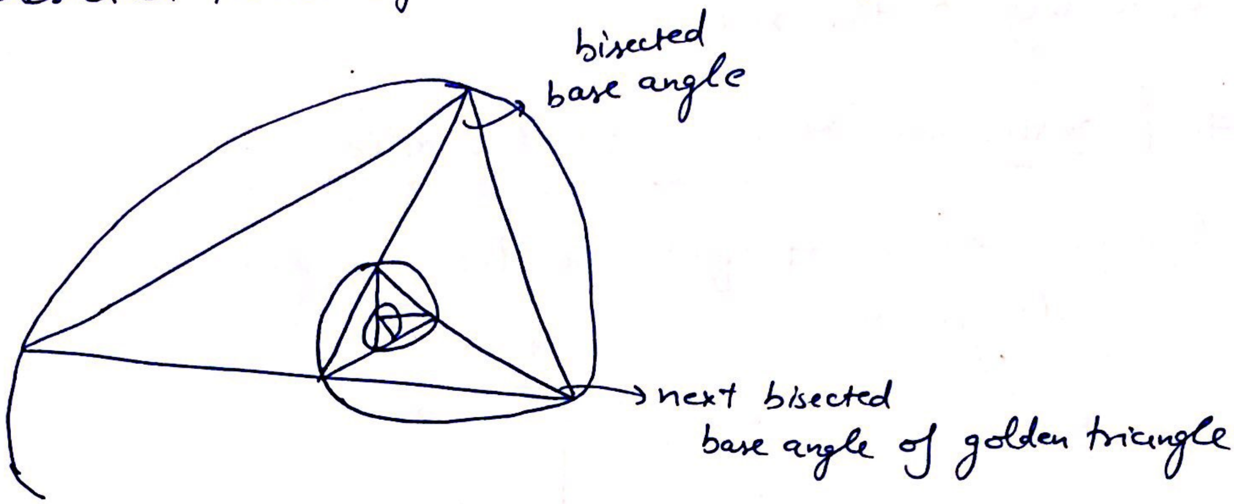
3) We now have

$$\frac{AB}{AC} = \frac{a}{\frac{a}{2}(1+\sqrt{5})} = \frac{\sqrt{5}-1}{2}$$

4.) The line segment AC is divided in the golden ratio by the point B.

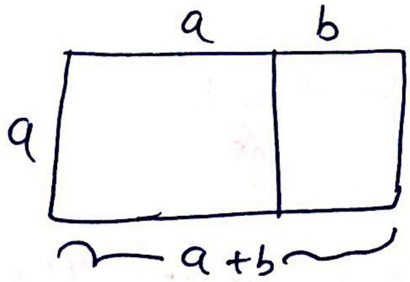
Golden Triangle: An ~~isso~~ isosceles triangle such that the ratio of the hypotenuse to the base is equal to the golden ratio. (its angles are 72°, 72°, 36°)

Now if a base angle is bisected, two isosceles triangles are generated, one of which is a new golden triangle.



Using this method, new golden triangles may be generated. Here, we obtain an equiangular or logarithmic spiral in the above generated triangles. (~~having 4a~~

Golden Rectangle: A golden rectangle is a rectangle whose side lengths are in the golden ratio which is 1.618...... i.e



$\cancel{a+b}$

$$\frac{a+b}{a} = \text{golden ratio}$$

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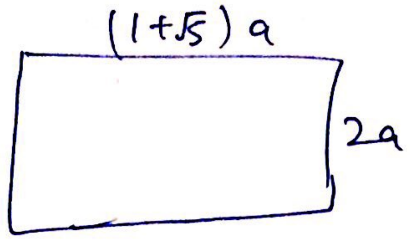
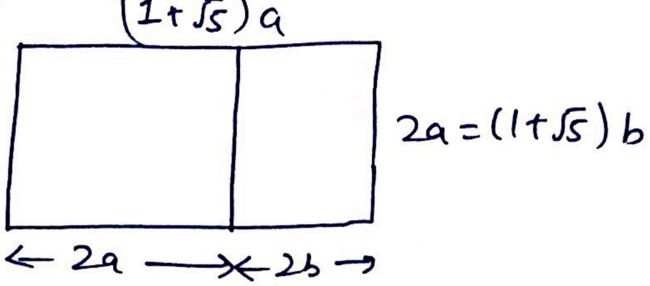
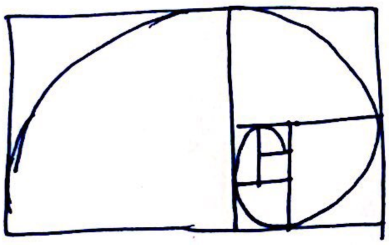


By removing from a golden rectangle a square, one side of which coincides with the shorter side of the original rectangle, we generate another golden rectangle



By this method new golden rectangles may be generated



Note: i) If arcs are drawn between nonadjacent corners of the square as shown above, we obtain an approximation of a logarithmic spiral

ii) The true logarithmic spiral does not touch the sides of the rectangles.

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